Perturbation Dynamics of the Infinite Dicke Model

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By means of operator algebraic methods the dynamics of the Dicke model is investigated in the limit where the number of the two-level atoms goes to infinity and the interaction strength remains on a finite level. The infinite atomic system is treated as a mean field quantum lattice system. It is shown that the limiting dynamics is essentially determined by the collective behaviour of the atoms. With Trotter's product formula and perturbation theoretical methods we obtain explicit expressions for the unitary time evolution operators in the uncoupled representation.

1. Introduction

The Dicke model consists of a large system of two-level atoms interacting with the radiation field, and goes back to the original work by Dicke [1]. In quantum optics the model usually is treated with finitely many atoms and finitely many modes of the electromagnetic field. Here we perform the infinite atom limit (thermodynamic limit) and include arbitrarily many modes of the field, which is quantized in an arbitrary region of the euclidean space $\mathbb{R}^3$. In the limit the coupling strength is rescaled on the level of finitely many atoms, which one may interpret as a weak coupling of a reservoir of macroscopically many atoms to the radiation field. The macroscopic preparation of the atomic reservoir is characterized by the (sharp) values of the cooperation and excitation degrees of the two-level atoms.

In [2] for the same problem Davies investigated a peculiar kind of infinite atom limit. Regarding each two-level atom as a spin-1/2-system, the limit is there performed along a sequence of eigenstates of the total angular momenta, which are selected according to the given limiting cooperation and excitation numbers. In the course of the limiting procedure the atomic degrees of freedom are eliminated and an irreversible dynamics for the restriction to the photon field is obtained.

Contrary to [2] we here consider the infinitely many two-level atoms as a system for its own and treat it as a mean-field quantum lattice system [3–5]. Its representation is selected according to the above macroscopic preparation, which is determined by the chosen cooperation and excitation numbers. In the infinite atom limit we now get the reversible dynamics of the total system, the infinitely many atoms plus the radiation, and not only the irreversible one for the photons. We also clarify in which sense in [2] the classical behaviour of the atoms comes into play. For our treatment of the dynamical problem we use methods totally different from those in [2]. In Section 4 we will return in some more detail to the relationship between [2] and the present work.

Having found the limiting dynamics it is possible to calculate the emitted radiation (cf. also [2, 6]). For an arbitrary (normal in the representation) initial state the radiation should become coherent for large times [7]. The coherence properties should be induced by the collective behaviour of the atoms (cooperation and excitation numbers). Such an investigation, however, is deferred to a subsequent work [8].

We now turn to a more detailed description of the model and the present work. For example let us consider the quantized electromagnetic field in the whole euclidean space $\mathbb{R}^3$. With the spin operators s_n^k, $k \in \{1, 2, 3, +, -\}$, for the n-th two-level atom the Dicke Maser Hamiltonian for N atoms in the rotating wave approximation is given by

$$H_N = \sum_{n=1}^{N} \varepsilon\, s_n^3 \otimes \mathbb{1} + \mathbb{1} \otimes H_s + \frac{\lambda}{N} \sum_{n=1}^{N} \left(s_n^- \otimes a^*(\phi) + s_n^+ \otimes a(\phi)\right). \quad (1.1)$$

Here $\varepsilon > 0$ is the level-splitting energy of each two-level atom. H_s, $a^*(\phi)$ and $a(\phi)$ denote the free Hamiltonian, the smeared creation resp. annihilation operators of the photon field in position space. In momentum space they are expressed by $\sum_\alpha \int |k|\, a_{k,\alpha}^* a_{k,\alpha}\, d^3k$,

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$\sum_\alpha \int \hat{\phi}_\alpha(k)\, a^*_{k,\alpha}\, d^3k$, resp. $\sum_\alpha \int \overline{\hat{\phi}_\alpha(k)}\, a_{k,\alpha}\, d^3k$, where $a^*_{k,\alpha}$ is the creation operator for the mode $k \in \mathbb{R}^3$ with polarisation α. The values $\hat{\phi}(k) \in \mathbb{C}^3$ express the coupling constants between each atom and the field mode $k \in \mathbb{R}^3$, they are calculated from the wavefunctions for the two energy levels of the considered type of atoms or molecules. ϕ is the Fourier inverse of the coupling function $\hat{\phi}: \mathbb{R}^3 \to \mathbb{C}^3$. It is $\nabla \cdot \phi = 0$.

In the Hamiltonians H_N, $N \in \mathbb{N}$, the weak coupling yields the factor N^{-1}, so that in the limit $N \to \infty$ the coupling strength remains on the level of $\lambda > 0$ atoms. This contrasts with [9, 10, 11], where the coupling is taken as $N^{-1/2}$.

In Sect. 2 we introduce the free infinite atomic system as a mean field quantum lattice system. The system is characterized by three independent macroscopic observables, which are given by the cooperation and excitation degrees, and some kind of collective phase. According to choosen sharp values of the (global) cooperation and excitation numbers, η resp. γ, we now motivate the representation $\Pi_a \equiv \Pi_a^{\eta,\gamma}$ of its associated C*-algebra. Then the collective behaviour of the atomic system in the representation Π_a is only expressed by the macroscopic phase angle $\vartheta \in [0, 2\pi[$. Further we discuss the limiting dynamics for the free atoms (due to "$\lim_{N\to\infty} \sum_{n=1}^{N} \varepsilon s_n^3$"), and in the dynamically invariant representation Π_a we state its generating Hamiltonian, which is the renormalized energy operator for the free atomic system.

Section 3 is devoted to the photon field. We start from an operator algebraic point of view and use the Weyl algebra as the photonic observables. For the relevant representation we choose the Fock representation Π_F, since we assume to be present only some few photons at time zero. However, our methods also work in other representations, e.g. in the temperature representations (cf. e.g. [12]), or in the representations over macroscopic coherent states [13, 14].

The limiting dynamics of the Dicke model (due to "$\lim_{N\to\infty} H_N$") is presented in Section 4. In terms of perturbation techniques we prove its existence in the representation $\Pi_a \otimes \Pi_F$. It follows that in the limit the photons are coupled to the classical part of the atomic system. By combining the perturbation expansion with Trotter's product formula the unitary time evolution operators are obtained in an explicit closed form. From these expressions the cocycle equations of Davies [2, 6] are deduced. In this way Davies' treatment of the model is incorporated into the general scheme of operator algebraic model studies.

In the Conclusions (Section 5) we shortly compare our techniques with somewhat related model discussions and with Dicke's intuitive idea about the origin of coherence.

2. The Atomic System

We give here a short self-contained description of the infinite atomic system and its free dynamics using the methods, which have been developed in [3, 4], and [5].

The two-level atoms are indexed by the natural numbers $n \in \mathbb{N}$. The set of all finite subsets of $\mathbb{N}$

$$\mathscr{L} := \{\Lambda \subset \mathbb{N} \,|\, |\Lambda| < \infty\},$$

where $|\Lambda|$ denotes the cardinality of Λ, is directed by inclusion. The observables of one atom consist of the lowering and rising operators $\sigma^\pm = \sigma^1 \pm i\sigma^2$ and the number operator σ^3, the algebraic span of which are all of the complex 2×2-matrices $\mathbb{M}_2$ (σ^k, $k = 1, 2, 3$, are the usual Pauli matrices). Thus the observables for the atoms in the finite lattice region $\Lambda \in \mathscr{L}$ are given by

$$\mathscr{A}(\Lambda) := \bigotimes_{n \in \Lambda} \mathbb{M}_2\,,$$

and for the infinite atomic system by (the C*-inductive limit [15])

$$\mathscr{A} := \bigotimes_{n \in \mathbb{N}} \mathbb{M}_2\,.$$

$\mathscr{A}$ is an antiliminary C*-algebra and has many inequivalent representations [16]. The local algebra $\mathscr{A}(\Lambda)$, $\Lambda \in \mathscr{L}$, are embedded into $\mathscr{A}$ by adjoining the unit $\mathbb{1}_2$ of $\mathbb{M}_2$ at the lattice points $n \notin \Lambda$.

The embedding of $\frac{1}{2}\sigma^k$ into the n-th factor of $\mathscr{A}$ is

$$s_n^k := \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2 \otimes \underbrace{\tfrac{1}{2}\sigma^k}_{\text{position } n} \otimes \mathbb{1}_2 \otimes \cdots \in \mathscr{A}, \qquad k \in \{1, 2, 3, +, -\}\,.$$

Regarding each two-level atom as a spin-1/2-system the total angular momentum observables and their densities for the atoms in the region $\Lambda \in \mathscr{L}$ are

$$J_\Lambda^k := \sum_{n \in \Lambda} s_n^k\,, \quad j_\Lambda^k := \frac{J_\Lambda^k}{|\Lambda|}\,,$$

with the vector squares

$$J_\Lambda^2 := (J_\Lambda^1)^2 + (J_\Lambda^2)^2 + (J_\Lambda^3)^2\,, \quad j_\Lambda^2 := \frac{J_\Lambda^2}{|\Lambda|^2}\,.$$

The Hamiltonian for the independent atoms with indices in $\Lambda \in \mathscr{L}$ is (after dropping a renormalization constant)

$$A_\Lambda = |\Lambda| \, \varepsilon j_\Lambda^3, \quad \Lambda \in \mathscr{L}, \tag{2.1}$$

where $\varepsilon > 0$ denotes the level-splitting energy. This and the atomic coupling operators $j_\Lambda^\pm$ in the interaction of (1.1) are invariant under permutations of the atomic numbering in Λ which is typical for mean field systems.

Kinematical Structure

The basic strategy to deal with such a system is to consider first not the whole of all states $\mathscr{S}(\mathscr{A}) \equiv \mathscr{S}$ of $\mathscr{A}$ but the set $\mathscr{S}^p(\mathscr{A}) \equiv \mathscr{S}^p$ of all permutation invariant states, which is a convex, weak*-compact subset with the extreme boundary $\partial_e \mathscr{S}^p$. The states in $\mathscr{S}^p$ represent the collective properties in an idealized manner, but the transition to the more realistic folium $\mathscr{F}^p \supset \mathscr{S}^p$, that is the smallest norm-closed split face of $\mathscr{S}$ which contains $\mathscr{S}^p$, is easily performed. We refer permanently to the subsequent results of Stormer [18], in the formulation of which we make use of the well-known affine homeomorphism between the states $\mathscr{S}(\mathbb{M}_2)$ on $\mathbb{M}_2$ and the ball $K_{1/2} := \{x \in \mathbb{R}^3 \mid \|x\| \le 1/2\}$ given by

$$\varrho \in \mathscr{S}(\mathbb{M}_2) \mapsto \left(\langle \varrho; \tfrac{1}{2}\sigma^1 \rangle, \langle \varrho; \tfrac{1}{2}\sigma^2 \rangle, \langle \varrho; \tfrac{1}{2}\sigma^3 \rangle\right) \in K_{1/2}.$$

The following results reveal $K_{1/2}$ to be the classical phase space of the atomic system, which expresses the collective behaviour of the infinitely many two-level atoms.

Proposition 2.1. *The following assertions are valid:*

(a) *For a state ω on $\mathscr{A}$ are equivalent*
  (i) $\omega \in \partial_e \mathscr{S}^p$;
  (ii) $\omega \in \mathscr{S}^p$ *and ω is factorial;*
  (iii) $\omega = \bigotimes_{n \in \mathbb{N}} \varrho$ *for some* $\varrho \in \mathscr{S}(\mathbb{M}_2)$.

(b) *The map* $\mathbb{P}: \partial_e \mathscr{S}^p \to \mathscr{S}(\mathbb{M}_2) = K_{1/2}$, $\bigotimes_{n \in \mathbb{N}} \varrho \mapsto \varrho$ *defines a $\sigma(\mathscr{A}^*, \mathscr{A}) - \|\cdot\|$-continuous homeomorphism.*

(c) *The map*

$$\mu \in M_+^1(K_{1/2}) \mapsto \int_{\partial_e \mathscr{S}^p} \varphi \, \mathrm{d}(\mu \circ \mathbb{P}(\varphi)) =: \mathbb{P}^{-1}(\mu) \in \mathscr{S}^p$$

defines a $\sigma(\mathscr{A}^*, \mathscr{A}) - \sigma(\mathscr{C}(K_{1/2})^*, \mathscr{C}(K_{1/2}))$-continuous, affine homeomorphism $\mathbb{P}$ from $\mathscr{S}^p$ onto the state space $\mathscr{S}(\mathscr{C}(K_{1/2})) = M_+^1(K_{1/2})$ (probability measures on $K_{1/2}$) of the continuous functions $\mathscr{C}(K_{1/2})$ on $K_{1/2}$. Moreover, the measure $\mathbb{P}(\omega) \circ \mathbb{P}$ is just the central measure of $\omega \in \mathscr{S}^p$, which is concentrated on $\partial_e \mathscr{S}^p$.

In the above sense the convex set $\mathscr{S}^p$ is affine-isomorphic to the Bauer simplex $M_+^1(K_{1/2})$, and hence the compact boundary set $\partial_e \mathscr{S}^p$ is just the point measures on $K_{1/2}$, that is, $K_{1/2}$ itself.

In the partially universal representation Π^p of $\mathscr{A}$ associated with $\mathscr{F}^p$ (consisting of the sum of GNS representations over the states in $\mathscr{F}^p$) the limits

$$j_k := \operatorname*{s-lim}_{\Lambda \in \mathscr{L}} \Pi^p(j_\Lambda^k), \quad k \in \{1, 2, 3, +, -\}, \tag{2.2}$$

exist in the strong operator topology and are elements of the center $\mathscr{Z}^p$ of the von Neumann algebra $\mathscr{M}^p := \Pi^p(\mathscr{A})''$. By [5] Theorem 2.1 there is a *-isomorphism $\mathscr{E}$ from $\mathscr{C}(K_{1/2})$ onto the smallest C*-algebra $\mathscr{N}^p \subset \mathscr{M}^p$ containing $\{j_1, j_2, j_3\}$, which carries over the coordinate functions $\pi_k: K_{1/2} \leftrightarrow \mathbb{R}$, $x \mapsto x_k$ to j_k for $k = 1, 2, 3$. $\mathscr{N}^p$ (resp. $\mathscr{C}^p(K_{1/2})$) determines the classical behaviour of the atomic system.

Every $\omega \in \mathscr{F}^p$ can be considered as a normal state on $\mathscr{M}^p$ (in the following we will do so), and thus by restriction as a state on $\mathscr{N}^p$, and hence $\omega \in \mathscr{F}^p$ induces via the *-isomorphism $\mathscr{E}^{-1}$ a state on $\mathscr{C}(K_{1/2})$ which we denote by $\nu_\omega \in M_+^1(K_{1/2})$. The support $\operatorname{supp}(\nu_\omega)$ of the measure ν_ω is by definition a closed subset of $K_{1/2}$. The classical state ν_ω is the probability distribution of the macroscopic parameters in $\omega \in \mathscr{F}^p$. If $X_G \in \mathscr{N}^p$ is the $\mathscr{E}$-image of $G \in \mathscr{C}(K_{1/2})$ then the associated expectation value for $\omega \in \mathscr{F}^p$ is given by the integral $\langle \omega; X_G \rangle = \int_{K_{1/2}} G(x) \, \mathrm{d}\nu_\omega(x)$, more specifically, if $P(j_1, j_2, j_3)$ is any polynomial in the (commuting) macroscopic operators $j_1, j_2, j_3 \in \mathscr{N}^p$ one has

$$\langle \omega; P(j_1, j_2, j_3) \rangle = \int_{K_{1/2}} P(x) \, \mathrm{d}\nu_\omega(x). \tag{2.3}$$

According to Proposition 2.1 it is $\nu_\omega = \mathbb{P}(\omega)$ for $\omega \in \mathscr{S}^p$.

Cooperation and Excitation Numbers

Let us consider a fixed $\Lambda \in \mathscr{L}$ and an arbitrary state ϱ on the local algebra $\mathscr{A}(\Lambda)$. Then commonly the cooperation number $\eta^\varrho \in [0, 1]$ associated with ϱ is defined by

$$\langle \varrho; \boldsymbol{J}_\Lambda^2 \rangle =: \frac{\eta^\varrho |\Lambda|}{2} \left(\frac{\eta^\varrho |\Lambda|}{2} + 1 \right).$$

η^ϱ determines the degree how strong the $|\Lambda|$ spins in the state ϱ are coupled in order to build up a "total angular momentum". The excitation number $\gamma^\varrho \in [0, 1]$ of ϱ,

$$\langle \varrho; J_\Lambda^3 \rangle =: \gamma^\varrho |\Lambda| - \frac{|\Lambda|}{2},$$

gives the ratio of the $|\Lambda|$ atoms in the upper level [1].

Now we extend the above notions to the infinite atomic system. We use the observable densities $\boldsymbol{j}_\Lambda^2$ and j_Λ^3, $\Lambda \in \mathscr{L}$, and define the *local* cooperation and excitation numbers, η_Λ^ω resp. γ_Λ^ω, of the state $\omega \in \mathscr{S}$ by

$$\langle \omega; \boldsymbol{j}_\Lambda^2 \rangle =: \frac{\eta_\Lambda^\omega}{2}\left(\frac{\eta_\Lambda^\omega}{2} + \frac{1}{|\Lambda|}\right), \quad \langle \omega; j_\Lambda^3 \rangle =: \gamma_\Lambda^\omega - \tfrac{1}{2}$$

for each $\Lambda \in \mathscr{L}$, and the *global* ones by the limits

$$\eta^\omega := \lim_{\Lambda \in \mathscr{L}} \eta_\Lambda^\omega, \quad \gamma^\omega := \lim_{\Lambda \in \mathscr{L}} \gamma_\Lambda^\omega, \tag{2.4}$$

which exist for each $\omega \in \mathscr{F}^p$ because of (2.2). The Cauchy-Schwarz inequality yields

$$|\gamma_\Lambda^\omega - \tfrac{1}{2}|^2 = \langle \omega; j_\Lambda^3 \rangle^2 \leq \langle \omega; (j_\Lambda^3)^2 \rangle \leq \langle \omega; \boldsymbol{j}_\Lambda^2 \rangle = \frac{\eta_\Lambda^\omega}{2}\left(\frac{\eta_\Lambda^\omega}{2} + \frac{1}{|\Lambda|}\right),$$

from which for the limits (2.4) follows $|\gamma^\omega - \frac{1}{2}| \leq \frac{\eta^\omega}{2}$. Here η^ω and γ^ω are given by the expectation values of the macroscopic operators $\boldsymbol{j}^2 = (j_1)^2 + (j_2)^2 + (j_3)^3 \in \mathscr{N}^p$ and $j_3 \in \mathscr{N}^p$ in the state ω (cf. (2.2) and (2.3)):

$$\langle \omega; \boldsymbol{j}^2 \rangle = \int_{K_{1/2}} \|x\|^2 \, dv_\omega(x) = \left(\frac{\eta^\omega}{2}\right)^2,$$

$$\langle \omega; j_3 \rangle = \int_{K_{1/2}} x_3 \, dv_\omega(x) = \gamma^\omega - \tfrac{1}{2}.$$

These considerations motivate a new parametrization of the classical phase space $K_{1/2}$ by means of the parameters cooperation η, excitation γ, and phase $\vartheta \in [0, 2\pi[$:

$$\eta := 2\|x\|, \quad \gamma := x_3 + \tfrac{1}{2},$$

$$\cos(\vartheta) := \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad \sin(\vartheta) := \frac{x_2}{\sqrt{x_1^2 + x_2^2}}. \tag{2.5}$$

Observe $\eta, \gamma \in [0, 1]$ with $|\gamma - \frac{1}{2}| \leq \frac{\eta}{2}$.

In the following let us consider some fixed, but arbitrary, parameters of cooperation and excitation, η resp. γ. Denote by $\mathscr{F}_{\eta,\gamma}$ the set of all states $\omega \in \mathscr{F}^p$ with sharp cooperation number $\eta^\omega = \eta$ and sharp excitation number $\gamma^\omega = \gamma$, that is, with vanishing variances

$$\langle \omega; (\boldsymbol{j}^2 - \langle \omega; \boldsymbol{j}^2 \rangle)^2 \rangle = 0, \quad \langle \omega; (j_3 - \langle \omega; j_3 \rangle)^2 \rangle = 0. \tag{2.6}$$

Lemma 2.2. *We have*

$$\mathscr{F}_{\eta,\gamma} = \{\omega \in \mathscr{F}^p \mid \operatorname{supp}(v_\omega) \subseteq T_{\eta,\gamma}\} \supset \mathbb{P}^{-1}(M_+^1(T_{\eta,\gamma})),$$

where $T_{\eta,\gamma} = \{x \in \mathbb{R}^3 \mid \|x\| = \eta/2, x_3 = \gamma - 1/2\} \subset K_{1/2}$, $\mathscr{F}_{\eta,\gamma}$ *is a subfolium of* $\mathscr{F}^p$ *which is maximal in the sense of* [5].

Proof: Consider an $\omega \in \mathscr{F}_{\eta,\gamma}$ and the Hilbert space $L^2(K_{1/2}, v_\omega)$ with unit vector $w(x) \equiv 1$. Using the *-isomorphism $\mathscr{E}$ and the coordinate function $\pi_3 : \mathbb{R}^3 \to \mathbb{R}$, $x \mapsto x_3$, (2.6) may be rewritten as

$$\|(\|\cdot\|^2 - (\eta/2)^2) w\|_{L^2}^2 = \int_{K_{1/2}} (\|x\|^2 - (\eta/2)^2)^2 \, dv_\omega(x) = 0,$$

$$\|(\pi_3 - (\gamma - 1/2)) w\|_{L^2}^2 = \int_{K_{1/2}} (x_3 - (\gamma - 1/2))^2 \, dv_\omega(x) = 0,$$

from which it follows that w is a simultaneous eigenvector for the multiplication operators $\|\cdot\|^2$ and π_3. We get $\|x\| = \eta/2$ and $x_3 = \gamma - \frac{1}{2}$ for v_ω-almost all $x \in K_{1/2}$ showing v_ω being concentrated on $T_{\eta,\gamma}$. Obviously $\mathbb{P}^{-1}(M_+^1(T_{\eta,\gamma})) \subset \mathscr{F}_{\eta,\gamma}$, and the assertion is proven. □

Dynamical Structure

The local Hamiltonians (2.1) generate the local Heisenberg dynamics on $\mathscr{A}$

$$\alpha_t^\Lambda(.) = \exp\{itA_\Lambda\} \cdot \exp\{-itA_\Lambda\}$$

for each $t \in \mathbb{R}$ and $\Lambda \in \mathscr{L}$. We have the following result.

Proposition 2.3. *There is a unique* C*-*dynamics* $(\alpha_t)_{t \in \mathbb{R}}$ *on* $\mathscr{A}$ *such that*

$$\alpha_t(X) = \|\cdot\|\text{-}\lim_{\Lambda \in \mathscr{L}} \alpha_t^\Lambda(X) \quad \forall t \in \mathbb{R} \quad \forall X \in \mathscr{A}.$$

The dual group $(\alpha_t^*)_{t \in \mathbb{R}}$ *leaves* $\mathscr{F}^p$, $\mathscr{F}_{\eta,\gamma}$, $\mathscr{S}^p$, *and* $\partial_e \mathscr{S}^p$ *invariant, and acts on* $\mathscr{S}^p$ *and* $\partial_e \mathscr{S}^p$ *as*

$$\alpha_t^*|_{\mathscr{S}^p} = \mathbb{P}^{-1} \circ \varphi_t^{**} \circ \mathbb{P}, \quad \alpha_t^*|_{\partial_e \mathscr{S}^p} = \mathbb{P}^{-1} \circ \varphi_t \circ \mathbb{P},$$

where $\varphi_t^{**} : M_+^1(K_{1/2}) \to M_+^1(K_{1/2})$, $\mu \mapsto \mu \circ \varphi_t^{-1}$ *with the flow* φ_t *on* $K_{1/2}$ *defined to be the rotation around the* x_3*-axis with velocity* ε, *that is*

$$\varphi_t(x) = \begin{pmatrix} \cos(\varepsilon t) & -\sin(\varepsilon t) & 0 \\ \sin(\varepsilon t) & \cos(\varepsilon t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x \in K_{1/2}.$$

Because of $\alpha_t^*(\mathscr{F}_{\eta,\gamma}) = \mathscr{F}_{\eta,\gamma}$ each α_t extends in the representation $\Pi_{\eta,\gamma}$ associated with the folium $\mathscr{F}_{\eta,\gamma}$ to a W*-automorphism $\tilde{\alpha}_t$ on $\mathscr{M}_{\eta,\gamma} = \Pi_{\eta,\gamma}(\mathscr{A})''$, which solves the problem of the free atomic limiting dynamics completely for the interesting states with sharp cooperation and excitation numbers. However, since $\|\alpha_t^*(\omega) - \omega\| = 2$ for $t \neq k2\pi$, $k \in \mathbb{Z}$, and $\omega = \mathbb{P}(x)$ with $x \in K_{1/2}$ being not on the x_3-axis, the group $\tilde{\alpha}_t$, $t \in \mathbb{R}$, is not a W*-dynamical system, and consequently not implementable by a unitary group on an appropriate representation Hilbert space (e.g. the standard representation), which is strongly continuous.

To overcome this difficulty we concentrate our further effort on the covariant subfolium $\mathscr{F}_a^{\eta,\gamma} \equiv \mathscr{F}_a$ of $\mathscr{F}_{\eta,\gamma}$ which is generated by the α_t^*-invariant state

$$\omega_a^{\eta,\gamma} \equiv \omega_a := \bar{\mathbb{P}}^{-1}(\lambda)\,, \tag{2.7}$$

where $\lambda \equiv \lambda^{\eta,\gamma}$ is the normalized Lebesgue measure on the circle line $T_{\eta,\gamma}$. λ is the point measure, if the radius of $T_{\eta,\gamma}$,

$$\beta^{\eta,\gamma} \equiv \beta := \sqrt{(\eta/2)^2 - (\gamma - 1/2)^2}\,, \tag{2.8}$$

is zero. Observe $\bar{\mathbb{P}}^{-1}(\mu) \in \mathscr{F}_a$ for each $\mu \in M_+^1(T_{\eta,\gamma})$ which is absolutely continuous with respect to λ, and which demonstrates the folium $\mathscr{F}_a$ being big enough for the relevant classical structure. $\mathscr{F}_a$ consists just of the normal states of the GNS representation $(\Pi_a, \mathscr{H}_a, \Omega_a)$ of ω_a.

By standard arguments there exists a unique self-adjoint operator A in $\mathscr{H}_a$ with

$$A\Omega_a = 0, \quad \Pi_a(\alpha_t(X)) = \mathrm{e}^{itA}\, \Pi_a(X)\, \mathrm{e}^{-itA} \quad \forall t \in \mathbb{R} \quad \forall X \in \mathscr{A}\,. \tag{2.9}$$

A obviously is the renormalized energy operator of the free atoms. To obtain more information about A we have to realize $(\Pi_a, \mathscr{H}_a, \Omega_a)$ explicitely. It is [19]

$$\mathscr{H}_a = \mathscr{H}_0 \otimes \mathrm{L}^2(T_{\eta,\gamma}, \lambda), \quad \Omega_a = \xi_0 \otimes w,$$
$$\Pi_a = \int_{\partial_e \mathscr{S}^p}^{\oplus} \Pi_\varphi \, \mathrm{d}(\lambda \circ \mathbb{P}(\varphi)) \tag{2.10}$$

with a separable infinite dimensional Hilbert space $\mathscr{H}_0$ and a unit vector $\xi_0 \in \mathscr{H}_0$, which follows from the fact that $\lambda \circ \mathbb{P}$ is the central measure of $\omega_a \in \mathscr{S}^p$ by Proposition 2.1 (w denotes the constant function $w(x) \equiv 1$). The associated von Neumann algebra and its center are given by

$$\mathscr{M}_a = \Pi_a(\mathscr{A})'' = \mathscr{M}_0 \mathbin{\bar{\otimes}} \mathrm{L}^\infty(T_{\eta,\gamma}, \lambda),$$
$$\mathscr{Z}_a = \mathscr{M}_a \cap \mathscr{M}_a' = \mathbb{1}_0 \otimes \mathrm{L}^\infty(T_{\eta,\gamma}, \lambda) \tag{2.11}$$

with the factorial von Neumann algebra $\mathscr{M}_0$ acting on $\mathscr{H}_0$ with the cyclic vector ξ_0. $\mathscr{M}_0$ is of type III_λ with $\lambda = \eta/2 + 1/2$ if $\eta \neq 1$ [17], and $\mathscr{M}_0 = \mathscr{B}(\mathscr{H}_0)$ for $\eta = 1$. The second factor in the tensor products (2.10) and (2.11) represent the classical structure of the atomic system with respect to the representation Π_a, resp. the folium $\mathscr{F}_a$.

Since Π_a is a subrepresentation of Π^p the local densities j_Λ^k, $\Lambda \in \mathscr{L}$, converge in the strong operator topology (cf. (2.2)), $\operatorname{s-lim}_{\Lambda \in \mathscr{L}} \Pi_a(j_\Lambda^k) = c(\Pi_a)\, j_k$, $k \in \{1, 2, 3, +, -\}$, where $c(\Pi_a) \in \mathscr{Z}^p$ is the central projector associated with the subrepresentation $\Pi_a \le \Pi^p$. In the above parametrization (2.5) of $K_{1/2}$ for $T_{\eta,\gamma}$ (remember that η, γ are fixed) only the phase angle $\vartheta \in [0, 2\pi[$ may vary, hence $\mathrm{L}^2(T_{\eta,\gamma}, \lambda) = \mathrm{L}^2([0, 2\pi[, \mathrm{d}\vartheta/2\pi)$, and one obtains with the radius (2.8)

$$B_\pm^{\eta;\lambda} \equiv B_\pm := c(\Pi_a)\, j_\pm = \mathbb{1}_0 \otimes \beta \exp\{\pm i\vartheta\}\,,$$
$$c(\Pi_a)\, j_3 = (\gamma - 1/2)\, \mathbb{1}_a\,. \tag{2.12}$$

With the above preparation the operator A is given by

Proposition 2.4. *For $X \in \mathscr{A}(\Lambda)$, $\Lambda \in \mathscr{L}$, and $F \in \mathscr{C}^1(T_{\eta,\gamma})$ we have $\Pi_a(X)\,(\mathbb{1}_0 \otimes F)\,\Omega_a \in \mathscr{D}(A)$; such elements form a core for A, and*

$$A(\Pi_a(X)(\mathbb{1}_0 \otimes F)\,\Omega_a) = [\Pi_a(A_\Lambda), \Pi_a(X)]\;(\mathbb{1}_0 \otimes F)\,\Omega_a + \Pi_a(X)\left(\mathbb{1}_0 \otimes \frac{\varepsilon}{i}\frac{\mathrm{d}F}{\mathrm{d}\vartheta}\right)\Omega_a\,.$$

Proof: [20], [19], cf. also [21]. □

As a consequence of Proposition 2.4 the unitary e^{itA} leaves the subspace $\xi_0 \otimes \mathrm{L}^2(T_{\eta,\gamma}, \lambda) \cong \mathrm{L}^2(T_{\eta,\gamma}, \lambda)$ of $\mathscr{H}_a$ invariant. The restriction of A to $\xi_0 \otimes \mathrm{L}^2(T_{\eta,\gamma}, \lambda)$ is just the differentiation operator

$$A(\xi_0 \otimes F) = \xi_0 \otimes \frac{\varepsilon}{i}\frac{\mathrm{d}F}{\mathrm{d}\vartheta} \qquad \forall F \in \mathscr{C}^1(T_{\eta,\gamma}) \tag{2.13}$$

with periodic boundary conditions, from which follows

$$\exp\{itA\}(\xi_0 \otimes F) = \xi_0 \otimes \exp\left\{it\frac{\varepsilon}{i}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right\}F = \xi_0 \otimes F \circ \varphi_t^{\eta,\gamma}\,.$$

Here $\varphi_t^{\eta,\gamma}$ denotes the restriction of the flow φ_t of Proposition 2.3 from $K_{1/2}$ to $T_{\eta,\gamma}$, and is given by

$$\varphi_t^{\eta,\gamma}(\vartheta) = \vartheta + \varepsilon t \qquad \forall t \in \mathbb{R} \quad \forall \vartheta \in [0, 2\pi[\,. \tag{2.14}$$

If $G \in L^\infty(T_{\eta,\gamma}, \lambda)$ denotes the multiplication operator with $\vartheta \mapsto G(\vartheta)$, that is $\mathbb{1}_0 \otimes G \in \mathscr{Z}_a$ by (2.11), then

$$\exp\{itA\}(\mathbb{1}_0 \otimes G)\exp\{-itA\} = \mathbb{1}_0 \otimes \exp\left\{it\frac{\varepsilon}{i}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right\} G \exp\left\{-it\frac{\varepsilon}{i}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right\} = \mathbb{1}_0 \otimes G \circ \varphi_t^{\eta,\gamma},$$

which together with (2.12) implies the essential relations for the collective raising and lowering operators

$$\exp\{itA\} B_\pm \exp\{-itA\} = \exp\{\pm it\varepsilon\} B_\pm \quad \forall t \in \mathbb{R}. \tag{2.15}$$

3. The Photon System

In our interpretation the radiation is supposed to consist of photons, but for notational simplicity and more general applications we let the emitted particles be bosons of unspecified nature.

The C*-algebra of the photon system is taken as the Weyl algebra $\mathscr{W}(E)$ over an appropriate testfunction space E – the one-photon resp. the one-boson space, [22]. E is a complex pre-Hilbert space with scalar product $\langle . | . \rangle$ linear in the second factor. $\mathscr{W}(E)$ is generated by nonzero elements $W(f)$, $f \in E$, – the Weyl operator – satisfying the Weyl relations

$$W(f)\,W(g) = \exp\left\{-\frac{i}{2}\operatorname{Im}\langle f | g\rangle\right\} W(f+g), \qquad W(f)^* = W(-f)$$

for all $f, g \in E$ [12, Theorem 5.2.8].

In a regular representation Π of $\mathscr{W}(E)$ the existence of selfadjoint field operators $\Phi_\Pi(f)$, $f \in E$, such that $\Pi(W(tf)) = \exp\{it\,\Phi_\Pi(f)\}$ $\forall t \in \mathbb{R}$, is ensured by Stone's theorem. The smeared annihilation and creation operators associated with Π

$$a_\Pi(f) := \frac{1}{\sqrt{2}}\big(\Phi_\Pi(f) + i\Phi_\Pi(if)\big), \qquad a_\Pi^*(f) := \frac{1}{\sqrt{2}}\big(\Phi_\Pi(f) - i\Phi_\Pi(if)\big),$$

are densely defined and closed. It is $a_\Pi(f)^* = a_\Pi^*(f)$. $f \mapsto a_\Pi(f)$ is antilinear and $f \mapsto a_\Pi^*(f)$ is linear and they fulfill the canonical commutation relations (CCR)

$$[a_\Pi(f), a_\Pi(g)] = [a_\Pi^*(f), a_\Pi^*(g)] = 0, \qquad [a_\Pi(f), a_\Pi^*(g)] = \langle f | g\rangle\,\mathbb{1} \quad \forall f, g \in E$$

on suitable dense domains, respectively [12].

Let S be the single particle Hamiltonian, that is, S is a selfadjoint operator acting in the Hilbert space $\bar{E}$ (the completion of E) such that $e^{itS}(E) \subseteq E$ $\forall t \in \mathbb{R}$. The free photon dynamics is then given by the group of C*-automorphisms γ_t, $t \in \mathbb{R}$, on the Weyl algebra $\mathscr{W}(E)$ in terms of the quasi-free Bogoliubov transformations [12]

$$\gamma_t(W(f)) = W(e^{itS} f) \quad \forall f \in E \quad \forall t \in \mathbb{R}. \tag{3.1}$$

Since $\| W(f) - W(g) \| = 2$ $\forall f \neq g$ there is no pointwise norm-continuity in the time t of the group $t \mapsto \gamma_t$, and hence it is no C*-dynamical system. Some weaker continuity properties are only obtained in suitable representations (cf. e.g. [22]).

As mentioned in the introduction for the present investigation we take the Fock representation $(\Pi_F, F_+(\bar{E}))$ of $\mathscr{W}(E)$. $F_+(\bar{E})$ is the Bose-Fock space over $\bar{E}$. We denote the usual smeared Fock-Weyl, -field, -creation and -annihilation operators by $W_F(f)$, $\Phi_F(f)$, $a_F^*(f)$ and $a_F(f)$, $f \in \bar{E}$, respectively. Clearly $\Pi_F(W(f)) = W_{\mathscr{F}}(f)$ $\forall f \in E$.

In the Fock representation the free photon dynamics is unitarily implementable by means of the common second quantization $\mathrm{d}\Gamma(S)$ of the one-photon Hamiltonian S [12] (implying continuity time for $t \mapsto \exp\{it\,\mathrm{d}\Gamma(S)\}$ with respect to the strong operator topology)

$$\Pi_F(\gamma_t(Y)) = \exp\{it\,\mathrm{d}\Gamma(S)\}\,\Pi_F(Y)\exp\{-it\,\mathrm{d}\Gamma(S)\} \quad \forall Y \in \mathscr{W}(E) \quad \forall t \in \mathbb{R}. \tag{3.2}$$

4. The Limiting Dynamics of the Dicke Model

For the atomic system we use here the representation Π_a on $\mathscr{H}_a$ of $\mathscr{A}$ from Sect. 2, which is associated with the fixed parameters of cooperation and excitation numbers, η resp. γ.

For the composite system the algebra of observables is the C*-tensor product $\mathscr{A} \otimes \mathscr{W}(E)$ (which is unique since $\mathscr{A}$ and $\mathscr{W}(E)$ are nuclear). In the representation $\Pi_a \otimes \Pi_F$ on the Hilbert space $\mathscr{H}_a \otimes \mathscr{F}_+(\bar{E})$ the local Hamiltonians for the Dicke model are given by [1]

[1] In the situation of (1.1) E is a dense subspace of $L^2_{\mathrm{div}}(\mathbb{R}^3, \mathbb{C}^3)$, the transversal vector fields $f\colon \mathbb{R}^3 \to \mathbb{C}^3$, that is $\nabla \cdot f = 0$ in the distributional sense, and $S = \sqrt{-\Delta}$ with the (selfadjoint) Laplacian $\Delta = \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2}$.

$$H_\Lambda = \underbrace{\Pi_a(A_\Lambda) \otimes \mathbb{1}_F + \mathbb{1}_a \otimes d\Gamma(S)}_{=: K_\Lambda} + \underbrace{\lambda\big(\Pi_a(j_\Lambda^-) \otimes a_F^*(\phi) + \Pi_a(j_\Lambda^+) \otimes a_F(\phi)\big)}_{=: P_\Lambda}$$

with the (arbitrary) coupling function $\phi \in \bar{E}$ and each $\Lambda \in \mathscr{L}$.

By Section 2 one expects that in the infinite atom limit $\Lambda \to \mathbb{N}$ the local operators H_Λ should converge in some sense to the operator

$$H = \underbrace{A \otimes \mathbb{1}_F + \mathbb{1}_a \otimes d\Gamma(S)}_{=: K} + \underbrace{\lambda\big(B_- \otimes a_F^*(\phi) + B_+ \otimes a_F(\phi)\big)}_{=: P}.$$

Before proving the limiting dynamics let us first give some results and Dyson expansions associated with our Hamiltonians $H_\Lambda = K_\Lambda + P_\Lambda$, $\Lambda \in \mathscr{L}$, and the "limiting" Hamiltonian $H = K + P$.

Proposition 4.1. *Let* $D := \bigcap_{\alpha \geq 1} \mathscr{D}(\mathbb{1}_a \otimes \alpha^N)$, *where N is the number operator in the Fock space $F_+(\bar{E})$. It follows for an arbitrary coupling function $\phi \in \bar{E}$:*

(i) *The operator H is essentially selfadjoint. $D \cap \mathscr{D}(K)$ is a core for H.*

(ii) *Define $U_t := e^{itK}$ and for $t \in \mathbb{R}$ and $\boldsymbol{t} \in \mathbb{R}^n$ let*

$$F_t^{(n)}(\boldsymbol{t}) := U_{t-t_1} P U_{t_1-t_2} P U_{t_2-t_3} P \ldots U_{t_{n-1}-t_n} P U_{t_n}.$$

Then $D \subseteq \mathscr{D}(F_t^{(n)}(\boldsymbol{t}))$ and $\mathbb{R}^n \ni \boldsymbol{t} \mapsto F_t^{(n)}(\boldsymbol{t})\,\psi$ is continuous for each $\psi \in D$. If $U_t^{(n)}\psi := i^n \int_{t_1=0}^{t} dt_1 \int_{t_2=0}^{t_1} dt_2 \ldots \int_{t_n=0}^{t_{n-1}} dt_n\, F_t^{(n)}(\boldsymbol{t})\,\psi$, then

$$e^{itH}\psi = U_t\psi + \sum_{n=1}^{\infty} U_t^{(n)}\psi \qquad \forall \psi \in D \quad \forall t \in \mathbb{R},$$

where the series converges in norm. Moreover $U_t^{(n)}(D) \subseteq D$ and $e^{itH}(D) \subseteq D$.

(iii) *For each $g \in \bar{E}$ we have $(\mathbb{1}_a \otimes W_F(g))(D) \subseteq D$. Defining $\tau_t(X) := e^{itK} X e^{-itK}$ and $\tau_t^P(X) := e^{itH} X e^{-itH}$ for $X = P$ or $X \in \mathscr{B}(\mathscr{H}_a \otimes F_+(\bar{E}))$ one has for each $\psi \in D$ and each $C \in \mathscr{B}(\mathscr{H}_a)$*

$$\tau_t^P(C \otimes W_F(g))\,\psi = \sum_{n=0}^{\infty} i^n \int_{t_1=0}^{t} dt_1 \ldots \int_{t_n=0}^{t_{n-1}} dt_n [\tau_{t_n}(P), [\ldots [\tau_{t_1}(P), \tau_t(C \otimes W_F(g))] \ldots]]\,\psi.$$

Moreover for $\psi \in D$ one has the estimation

$$\| [\tau_{t_n}(P), [\ldots [\tau_{t_1}(P), \tau_t(C \otimes W_F(g))] \ldots]]\,\psi \| \leq (2^{9/2}\lambda \,\|\phi\|)^n \sqrt{n!}\, \|C\| \left(\sum_{k=0}^{\infty} \frac{(2^{7/2}\|g\|)^k}{\sqrt{k!}}\right) \cdot \|(\mathbb{1}_a \otimes 2^{3N/2})\psi\|.$$

(iv) *The results of* (i), (ii) *and* (iii) *are also valid for arbitrary $\Lambda \in \mathscr{L}$, if we replace H by H_Λ, K by K_Λ, and P by P_Λ. Especially the estimation of* (iii) *is uniform in $\Lambda \in \mathscr{L}$. However, if $\phi \in \mathscr{D}(S^{-1/2})$ we have a stronger result than* (i). *Then the local operators H_Λ, $\Lambda \in \mathscr{L}$, are selfadjoint, which is shown by a Kato-Rellich argument considering the interaction as a relatively bounded perturbation* [23], [24] *Lemma 4.1.*

Proof: The proof is similar to [25, Theorem 5.2]. Observe $\| B_\pm \| = \beta \leq 1$ and $\| j_\Lambda^\pm \| \leq 1$ $\forall \Lambda \in \mathscr{L}$. □

Theorem 4.2. *For the Heisenberg dynamics it holds in the strong operator topology*

$$\operatorname*{s-lim}_{\Lambda \in \mathscr{L}} e^{itH_\Lambda} Z e^{-itH_\Lambda} = e^{itH} Z e^{-itH}$$

for all $t \in \mathbb{R}$ and each $Z \in (\Pi_a \otimes \Pi_F)(\mathscr{A} \otimes \mathscr{W}(E))$.

Proof: Since the estimates in Proposition 4.1 (iii) are uniform in $\Lambda \in \mathscr{L}$ and $\operatorname*{s-lim}_{\Lambda \in \mathscr{L}} \Pi_a(\alpha_t^\Lambda(X)) = e^{itA}\,\Pi_a(X) \cdot e^{-itA}$ $\forall X \in \mathscr{A}$ by Proposition 2.3 and (2.9) it is immediate to check

$$\|\cdot\|\text{-}\lim_{\Lambda \in \mathscr{L}} e^{itH_\Lambda}(C \otimes W_F(g))\, e^{-itH_\Lambda}\psi = e^{itH}(C \otimes W_F(g))\, e^{-itH}\psi \qquad \forall t \in \mathbb{R} \quad \forall \psi \in D.$$

D being dense in $\mathscr{H}_a \oplus F_+(\bar{E})$ and $\mathsf{LH}\{C \otimes W_F(g) \mid C \in \mathscr{A}, g \in E\}$ being norm-dense in $(\Pi_a \oplus \Pi_F)(\mathscr{A} \otimes \mathscr{W}(E))$ the assertion follows. □

In the following we derive a closed form for the unitary group $(e^{itH})_{t \in \mathbb{R}}$. The ultimate reason for this possibility lies in the simple relation (2.15) for the operators $B_\pm = B_\mp^*$, which are elements of the center $\mathscr{Z}_a$ from (2.11). In [2, § 3] the restricted problem is solved by cocycle techniques. Here we proceed in a total different and more direct way and use a convergent Dyson expansion of e^{itH} and Trotter's product formula. We rewrite the representation Hilbert space $\mathscr{H}_a \otimes F_+(\bar{E})$ as a direct integral [26], [27], [28]

$$\mathscr{H}_a \otimes F_+(\bar{E}) = \mathscr{H}_0 \otimes \mathrm{L}^2([0, 2\pi[, \mathrm{d}\vartheta/2\pi) \otimes F_+(\bar{E}) = \mathscr{H}_0 \otimes \int_{[0,2\pi[}^{\oplus} F_+(\bar{E}) \frac{\mathrm{d}\vartheta}{2\pi}.$$

Based on this isomorphism we obtain the second main result of the present paper.

Theorem 4.3. *The unitaries* e^{itH} *are given for each* $t \in \mathbb{R}$ *by*

$$\mathrm{e}^{itH} = \left[\mathbb{1}_{\mathscr{H}_0} \otimes \int_{[0,2\pi[}^{\oplus} \mathrm{e}^{i\varkappa(t)} W_{\mathrm{F}}(f(t,\vartheta)) \frac{\mathrm{d}\vartheta}{2\pi}\right] [\mathrm{e}^{itA} \otimes \mathrm{e}^{it\mathrm{d}\Gamma(S)}],$$

where

$$\varkappa(t) = -\beta^2 \lambda^2 \operatorname{Im}\langle \phi | \Psi_t(S-\varepsilon)\phi\rangle = -\beta^2\lambda^2 \int_{u=0}^{t} \mathrm{d}u \int_{v=0}^{u} \mathrm{d}v \operatorname{Im}\langle \mathrm{e}^{i(S-\varepsilon)v}\phi | \mathrm{e}^{i(S-\varepsilon)u}\phi\rangle \in \mathbb{R},$$

$$f(t,\vartheta) = -\sqrt{2}\beta\lambda\,\mathrm{e}^{-i\vartheta}\,\Theta_t(S-\varepsilon)\phi = \sqrt{2}\beta\lambda\,\mathrm{e}^{-i\vartheta}\int_{u=0}^{t} \mathrm{d}u\,\mathrm{e}^{i(S-\varepsilon)u}\phi \in \bar{E},$$

with the bounded functions on the real line

$$\Psi_t(y) = \frac{\mathrm{e}^{ity} - 1 - ity}{(iy)^2}, \quad \Theta_t(y) = \frac{\mathrm{e}^{ity}-1}{iy}, \quad y \in \mathbb{R}.$$

Proof: For the proof we need several steps.

(i) Observing (2.12), (2.15) and $\mathrm{e}^{it\mathrm{d}\Gamma(S)} a_{\mathrm{F}}^{\#}(g)\,\mathrm{e}^{-it\mathrm{d}\Gamma(S)} = a_{\mathrm{F}}^{\#}(\mathrm{e}^{itS} g)$, on the elements from D we obtain with the expansion of Proposition 4.1 (ii)

$$\begin{aligned}
\mathrm{e}^{itH}\,\mathrm{e}^{-itK} &= \sum_{n=0}^{\infty} i^n \int_{t_n=0}^{t} \mathrm{d}t_n \ldots \int_{t_1=0}^{t_2} \mathrm{d}t_1 \left\{\prod_{\nu=1}^{n} \mathrm{e}^{it_\nu K} P\,\mathrm{e}^{-it_\nu K}\right\} \\
&= \sum_{n=0}^{\infty} i^n \int_{t_n=0}^{t} \mathrm{d}t_n \ldots \int_{t_1=0}^{t_2} \mathrm{d}t_1 \left\{\prod_{\nu=1}^{n} \lambda\big(\mathbb{B}_- \otimes a_{\mathrm{F}}^{*}(\mathrm{e}^{it_\nu(S-\varepsilon)}\phi) + B_+ \otimes a_{\mathrm{F}}(\mathrm{e}^{it_\nu(S-\varepsilon)}\phi)\big)\right\} \\
&= \sum_{n=0}^{\infty} i^n \int_{t_n=0}^{t} \mathrm{d}t_n \ldots \int_{t_1=0}^{t_2} \mathrm{d}t_1 \prod_{\nu=1}^{n} \mathbb{1}_0 \otimes \int_{[0,2\pi[}^{\oplus} \frac{\mathrm{d}\vartheta}{2\pi} \Phi_{\mathrm{F}}(\sqrt{2}\beta\lambda\,\mathrm{e}^{-i\vartheta}\,\mathrm{e}^{it_\nu(S-\varepsilon)}\phi) \\
&= \mathbb{1}_0 \otimes \int_{[0,2\pi[}^{\oplus} \frac{\mathrm{d}\vartheta}{2\pi} \sum_{n=1}^{\infty} i^n \int_{t_n=0}^{t} \mathrm{d}t_n \ldots \int_{t_1=1}^{t_2} \mathrm{d}t_1 \left(\prod_{\nu=1}^{n} \mathrm{e}^{it_\nu \mathrm{d}\Gamma(S-\varepsilon)} \Phi_{\mathrm{F}}(\sqrt{2}\lambda\beta\,\mathrm{e}^{-i\vartheta}\phi)\,\mathrm{e}^{-it_\nu\mathrm{d}\Gamma(S-\varepsilon)}\right) \\
&= \mathbb{1}_0 \otimes \int_{[0,2\pi[}^{\oplus} \frac{\mathrm{d}\vartheta}{2\pi} \exp\{it(\mathrm{d}\Gamma(S-\varepsilon) + \Phi_{\mathrm{F}}(\sqrt{2}\lambda\beta\,\mathrm{e}^{-i\vartheta}\phi))\}\,\mathrm{e}^{-it\mathrm{d}\Gamma(S-\varepsilon)}.
\end{aligned}$$

(ii) Here we calculate the term $\exp\{it(\mathrm{d}\Gamma(T) + \Phi_{\mathrm{F}}(g))\}$ by use of Trotter's product formula [29, Theorem VIII.31] (T is an arbitrary selfadjoint operator on $\bar{E}$ and $g \in \bar{E}$), which yields in the strong operator topology (observe $\exp\{i\Phi_{\mathrm{F}}(f)\} = W_{\mathrm{F}}(f)$)

$$\exp\{it(\mathrm{d}\Gamma(T) + \Phi_{\mathrm{F}}(g))\} = \operatorname*{s-lim}_{n\to\infty} \big(\exp\{i(t/n)\,\mathrm{d}\Gamma(S)\}\,W_{\mathrm{F}}((t/n)\,g)\big)^n. \tag{4.1}$$

We first calculate the right hand side for finite $n \in \mathbb{N}$ and then turn to the limit $n \to \infty$. By (3.1) and (3.2) we have $\exp\{i\tau\,\mathrm{d}\Gamma(T)\}\,W_{\mathrm{F}}(h)\exp\{-i\tau\,\mathrm{d}\Gamma(T)\} = W_{\mathrm{F}}(\mathrm{e}^{i\tau T} g)$. Thus with the Weyl relations we get

$$\begin{aligned}
[\exp\{i(t/n)\,\mathrm{d}\Gamma(T)\}\,W_{\mathrm{F}}((t/n)g)]^n &= \left[\prod_{l=1}^{n} \exp\{il(t/n)\,\mathrm{d}\Gamma(T)\}\,W_{\mathrm{F}}((t/n)g)\exp\{-il(t/n)\,\mathrm{d}\Gamma(T)\}\right][\exp\{it\,\mathrm{d}\Gamma(T)\}] \\
&= \left[\prod_{l=1}^{n} W_{\mathrm{F}}((t/n)\,\mathrm{e}^{il(t/n)T} g)\right][\exp\{it\,\mathrm{d}\Gamma(T)\}] \\
&= \exp\left\{-\frac{i}{2} t^2 \operatorname{Im}\langle g | \psi_t^{(n)}(T) g\rangle\right\} W_{\mathrm{F}}(t\,\phi_t^{(n)}(T) g)\,[\exp\{it\,\mathrm{d}\Gamma(T)\}]
\end{aligned}$$

with the functions

$$\psi_t^{(n)}(y) := \frac{1}{n^2} \sum_{l=1}^{n-1} \sum_{r=1}^{l} e^{ir(t/n)y}$$

and

$$\phi_t^{(n)}(y) := \frac{1}{n} \sum_{l=1}^{n} e^{il(t/n)y}, \quad y \in \mathbb{R}.$$

Obviously $\|\psi_t^{(n)}\|_\infty \leq \frac{1}{2}$ and $\|\phi_t^{(n)}\|_\infty \leq 1 \; \forall n \in \mathbb{N}$. Moreover

$$\lim_{n\to\infty} \psi_t^{(n)}(y) = \frac{\Psi_t(y)}{t^2}, \quad \lim_{n\to\infty} \phi_t^{(n)}(y) = \frac{\Theta_t(y)}{t} \quad \forall y \in \mathbb{R},$$

from which with the spectral calculus follows

$$\lim_{n\to\infty} \| t\,\phi_t^{(n)}(T)g - \Theta_t(T)g \| = 0$$

and similar for $\psi_t^{(n)}(T)$. Hence, using the continuity of $f \in \bar{E} \mapsto W_F(f)$ in the norm and the strong operator topology, [12, Proposition 5.2.4(4)], it follows from (4.1)

$$\exp\{it(\mathrm{d}\Gamma(T) + \Phi_F(g))\}$$

$$= \exp\left\{-\frac{i}{2}\,\mathrm{Im}\langle g \mid \Psi_t(T)g\rangle\right\} W_F(\Theta_t(T)g) \cdot \exp\{it\,\mathrm{d}\Gamma(T)\}.$$

(iii) Inserting the result of (ii) into the one of (i) yields the assertion. The integral expressions for $\varkappa(t)$ and $f(t, \vartheta)$ follow by direct calculations. □

A first link with [2] is given by the following corollary, which one easily checks by use of the integral formulas of $\varkappa(t)$ and $f(t, \vartheta)$ of the above Theorem.

Corollary 4.4. *The functions* $\varkappa\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto \varkappa(t)$ *and* $f\colon \mathbb{R} \times [0, 2\pi[\to \bar{E}$, $(t, \vartheta) \mapsto f(t, \vartheta)$ *from Theorem 4.3 satisfy the cocycle equations*

$$f(s+t, \vartheta) = f(s, \vartheta) + e^{isS} f(t, \varphi_s^{\eta,\gamma}(\vartheta)),$$

$$\varkappa(s+t) = \varkappa(s) + \varkappa(t) - \tfrac{1}{2}\,\mathrm{Im}\langle f(s, \vartheta) \mid e^{isS} f(t, \varphi_s^{\eta,\gamma}(\vartheta))\rangle$$

for all $\vartheta \in [0, 2\pi[$ *and each* $s, t \in \mathbb{R}$, *where* $\varphi_t^{\eta,\gamma}(\vartheta) = \vartheta + \varepsilon t$ *is the flow of* (2.14).

In order to give the link to the results of [2], especially Theorem 3.8, we restrict the unitary dynamics $(e^{itH})_{t\in\mathbb{R}}$, which is originally formulated on $\mathscr{H}_a \otimes F_+(\bar{E}) = \mathscr{H}_0 \otimes L^2([0, 2\pi[, \mathrm{d}\vartheta/2\pi) \otimes F_+(\bar{E})$ to the subspace

$$\xi_0 \otimes L^2([0, 2\pi[, \mathrm{d}\vartheta/2\pi) \otimes F_+(\bar{E}) \cong L^2([0, 2\pi[, \mathrm{d}\vartheta/2\pi) \otimes F_+(\bar{E}) =: \mathscr{K},$$

where ξ_0 is from (2.10).

Proposition 4.5. *For each* $t \in \mathbb{R}$ *the unitary* e^{itH} *leaves the subspace* $\mathscr{K}$ *of* $\mathscr{H}_a \otimes F_+(\bar{E})$ *invariant. The generator* $\tilde{H}$ *on* $\mathscr{K}$ *of the restricted strongly continuous group* $(e^{itH}|_{\mathscr{K}})_{t\in\mathbb{R}}$ *is given by*

$$\tilde{H} = \frac{\varepsilon}{i}\frac{\mathrm{d}}{\mathrm{d}\vartheta} \otimes \mathbb{1}_F + \mathbb{1}_{L^2} \otimes \mathrm{d}\Gamma(S) + \lambda\beta\left(e^{-i\vartheta} \otimes a_F^*(\phi) + e^{i\vartheta} \otimes a_F(\phi)\right),$$

where β *is the radius of* $T_{\eta,\gamma}$ *from* (2.8), *and the* $e^{\pm i\vartheta}$ *are considered as multiplication operators with the functions* $\vartheta \mapsto e^{\pm i\vartheta}$.

Proof: This is an immediate consequence of Proposition 2.4, the formulas (2.12) and (2.13), and Theorem 4.3. □

Obviously Proposition 4.5 states the connection of the present investigation and [2], [6] mentioned in the Introduction. It is seen that by the methods of [2] only the classical part of the atomic mean field system is grasped in a somewhat ad hoc way.

From Proposition 4.5 the irreversible limiting dynamics $T_t(\varrho)$ for the photons of [2, Eq. (1.8)] is obtained in the present context by the operations

$$T_t(\varrho) = \tilde{M}[e^{it\tilde{H}}(|w\rangle\langle w| \otimes \varrho)\,e^{-it\tilde{H}}] = M[e^{itH}(|\Omega_a\rangle\langle\Omega_a| \otimes \varrho)\,e^{-itH}],$$

where ϱ is an arbitrary density operator on the Fock space $F_+(\bar{E})$, $\Omega_a = \xi_0 \otimes w$ is the cyclic vector (2.10) of the state ω_a from (2.7). $\tilde{M}$ denotes the partial trace from $\mathrm{Tr}_+^1(\mathscr{K})$ onto $\mathrm{Tr}_+^1(F_+(\bar{E}))$, whereas M is those from $\mathrm{Tr}_+^1(\mathscr{H}_a \otimes F_+(\bar{E}))$ onto $\mathrm{Tr}_+^1(F_+(\bar{E}))$ averaging out the atomic states.

5. Conclusions

The above model discussion belongs to the rare non-trivial examples of a convergent perturbation theory for Bosons interacting with matter. Thus the treatment is naturally divided into three parts:

(1) The atomic system is realized in the representation, which belongs to the sharp preparation of the cooperation and excitation numbers. The weak clo-

sure of the represented atomic algebra is shown to have a non-trivial center consisting of the functions of the macroscopic phase angle. The macroscopic phase rotates with a velocity which is given by the atomic level-splitting.

(2) The free Boson system is realized in the Fock representation and obeys a quasi-free dynamics. In this vacuum sector the number of Bosons is smaller than macroscopic.

(3) The coupled system is realized in the product representation of the free systems. This is only possible by the N^{-1}-scaling of the coupling, where N is the number of the atoms. In the limit $N \to \infty$ this scaling renders the atomic coupling operators to (bounded) central elements. Since the Bosonic coupling operators are unbounded, the convergence of the perturbation series is not quite trivial and rests on results of [25], [24]. The coupling is still strong enough to shift the atomic collective ordering structure to the Bosons. In a subsequent paper [8] we show that the time asymptotic Boson states constitute a coherent radiation with maximal spectral density at the unrenormalized resonance frequency.

The closed expressions for the unitary translation operators are here a consequence of the thermodynamic limit. For finite atomic systems the perturbational treatment would be also possible by means of the techniques used in the present investigation, resp. in [25], [24]. But only in the simplest case of one atom with level-splitting zero one obtains closed expressions for the unitary time translations [30–32] (where in the two latter references one treats the (bounded) free atomic Hamiltonian as a "perturbation").

Some aspects of the infinite Dicke model with the $N^{-1/2}$-scaling were elaborated in [9]. As far as we know, the asymptotic radiation of this model has not yet been evaluated, and it is not clear, which of the scalings fits better to the physical effect of superradiance [33]. In the present investigation, which in contrast to [2] treats both subsystems on the same level, Dicke's original idea about the origin of coherence is confirmed in a concise manner: in certain preparations the macroscopic atomic system exhibits collective variables, especially a macroscopic phase. This aspect enters the theoretical description here by the choice of the representation of the atomic observable algebra. Every atomic initial state of this representation may have arbitrarily many uncorrelated one particle phases but is nevertheless a relatively small deviation of a collectively ordered state. The interaction shifts the macroscopic phase to the photon side. More precisely, the exact model study of the present and the subsequent paper [8] gives a detailed picture how the collective atomic ordering influences the radiation dynamics ending up with a coherent photon state.

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